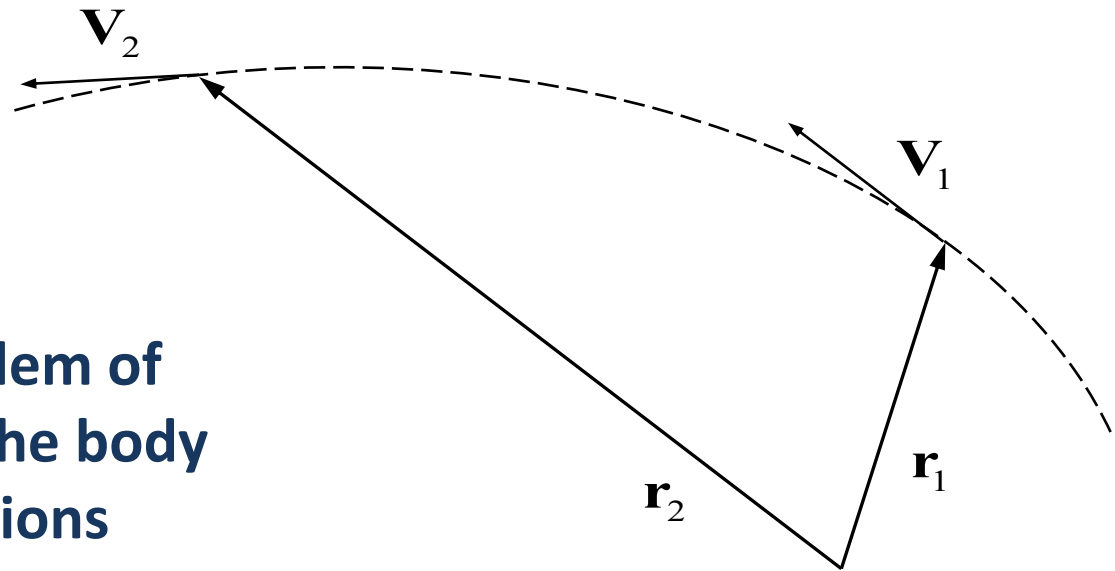
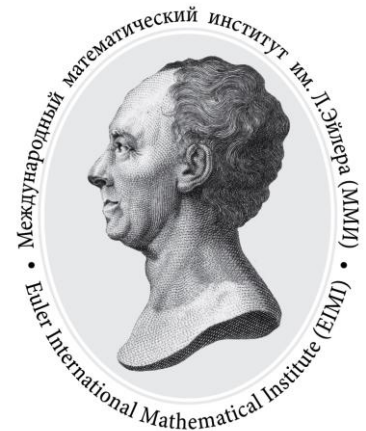


The Euler-Lambert problem of determining the orbit of the body based on its two positions



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The Lambert Problem and Its Application

A classical problem of celestial mechanics: determining the orbit of a celestial body's motion from two observations (a boundary-value problem):

$$\mathbf{r}(t_1) = \mathbf{r}_1, \quad \mathbf{r}(t_2) = \mathbf{r}_2, \quad \Delta t = t_2 - t_1,$$

$$\mathbf{v}(t_1) = ? \quad \mathbf{v}(t_2) = ?$$

In spaceflight mechanics, it allows one to determine the characteristic-velocity expenditure for a two-impulse transfer:

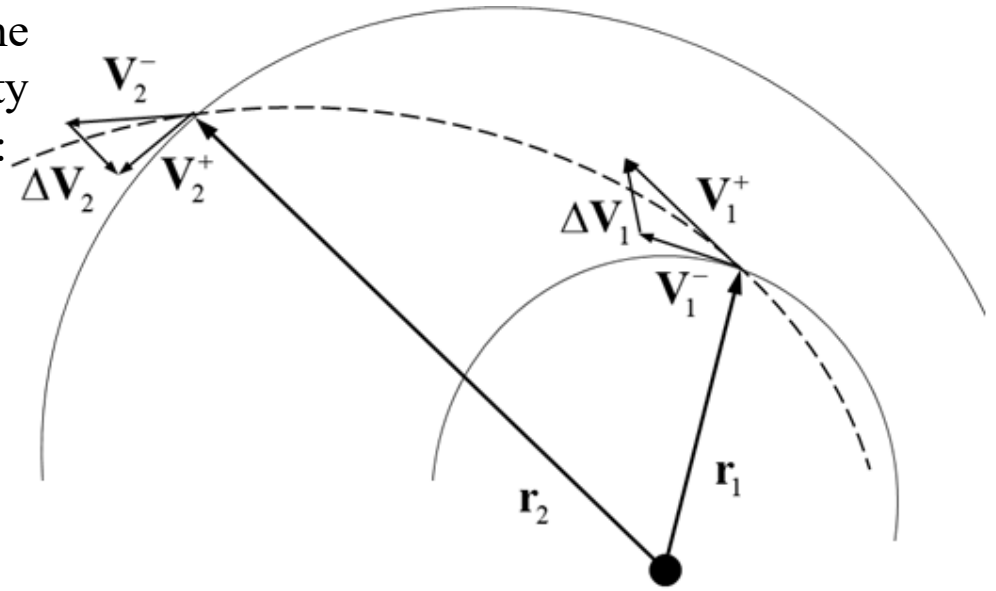
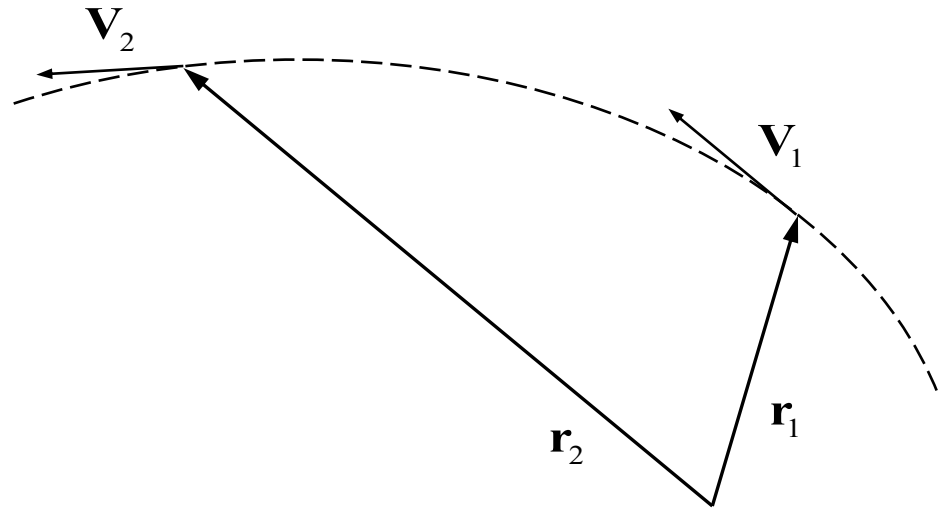
$$\mathbf{r}(t_1) = \mathbf{r}_1, \quad \mathbf{v}(t_1) = \mathbf{v}_1^-,$$

$$\mathbf{r}(t_2) = \mathbf{r}_2, \quad \mathbf{v}(t_2) = \mathbf{v}_2^+,$$

$$\Delta t = t_2 - t_1,$$

$$\Delta \mathbf{v}_1 = \mathbf{v}_1^+ - \mathbf{v}_1^-, \quad \Delta \mathbf{v}_2 = \mathbf{v}_2^+ - \mathbf{v}_2^-,$$

$$\Delta \mathbf{v}_\Sigma = |\Delta \mathbf{v}_1| + |\Delta \mathbf{v}_2|.$$



General Form of the Solution and Representations

Since the **two-body problem** admits an **analytical solution**, the boundary-value problem reduces to a single **nonlinear equation** relating the transfer time to the orbital parameters.

This equation follows from **Kepler's equation** and can be represented in various forms:

- based on **orbital elements**,
- using a **universal variable**,
- through **regularized variables** (KS variables),
- or via solutions of the **inverse dynamics problem**.

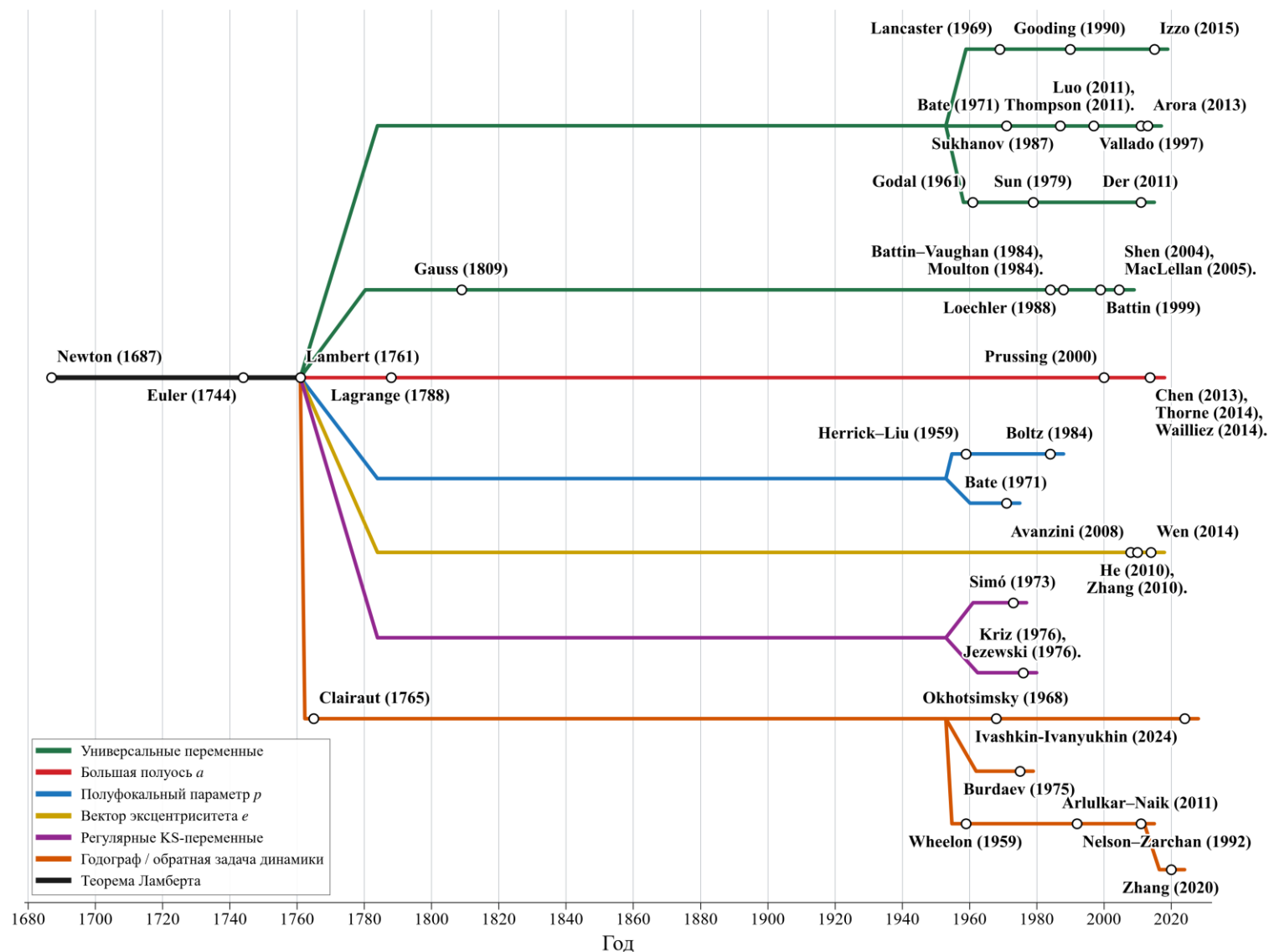
$$\mathbf{r}_2 = f(x)\mathbf{r}_1 + g(x)\mathbf{v}_1,$$

$$\mathbf{v}_2 = \dot{f}(x)\mathbf{r}_1 + \dot{g}(x)\mathbf{v}_1,$$

$$\Delta t = F(\mathbf{r}_1, \mathbf{r}_2, x)$$

where Δt – transfer time, $\mathbf{r}_1, \mathbf{r}_2$ – initial and final position, $\mathbf{v}_1, \mathbf{v}_2$ – initial and final velocity on the transfer trajectory, x – parameter of the transfer orbit.

Development of Solution Methods and Representations

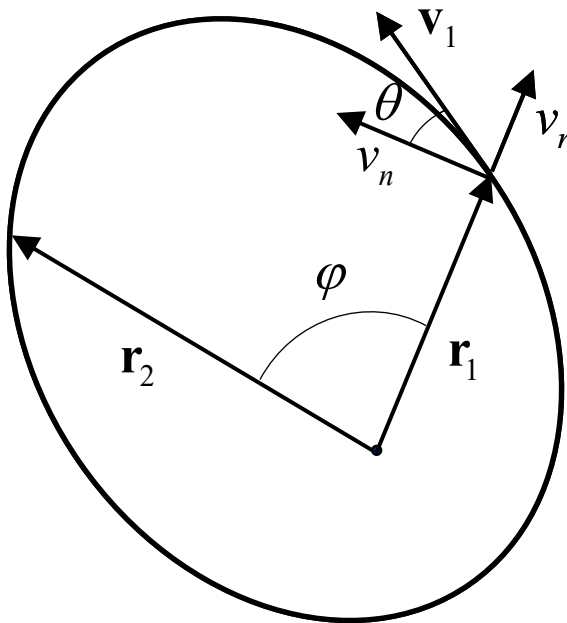


Okhotsimsky's Dynamical Method of Solution

Solution of Clairaut's equation in Okhotsimsky's form (for the inverse problem of ballistics):

$$u = \frac{1}{r}, \quad \mathbf{c} = \mathbf{r} \times \mathbf{v} \rightarrow v_r = \frac{dr}{dv} \frac{dv}{dt} = -c \frac{du}{dv}, \quad v_n = r \frac{dv}{dt} = cu,$$

$$h = v^2 - \frac{2\mu}{r} \rightarrow h = c^2 \left(\left(\frac{du}{dv} \right)^2 + u^2 \right) - 2\mu u, \quad \frac{dh}{dv} = 0 \quad \frac{du}{dv} \neq 0 \Rightarrow$$



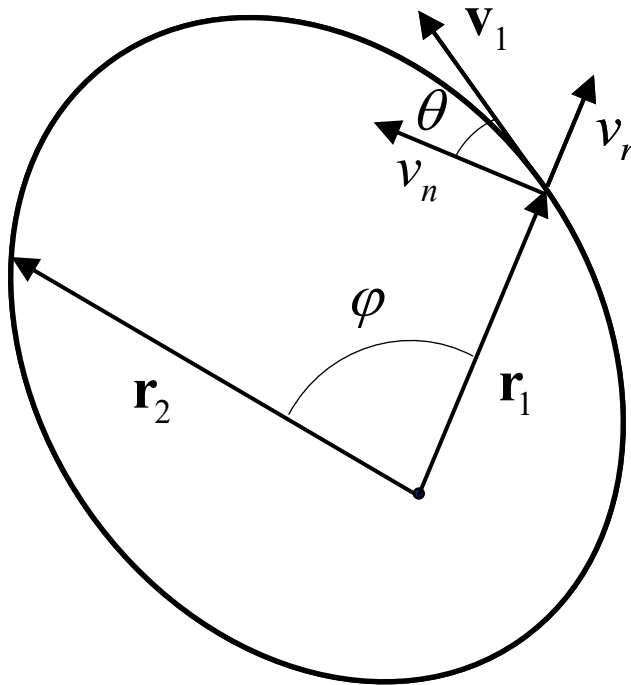
$$\frac{d^2 u}{d\varphi^2} + u = \frac{\mu}{c^2}$$

$$u(\varphi) = \frac{\mu}{c^2} + c_1 \cos(\varphi) + c_2 \sin(\varphi),$$

$$c_1 = \frac{1}{r_1} - \frac{\mu}{c^2}, \quad c_2 = \frac{\operatorname{tg}(\theta)}{r_1}.$$

Okhotsimsky's Dynamical Method of Solution

Solution of Clairaut's equation in Okhotsimsky's form (for the inverse problem of ballistics):



$$v_1 = \sqrt{\frac{\mu r_2 (1 - \cos(\varphi))}{r_1 (r_1 \cos^2(\theta) - r_2 \cos(\varphi + \theta) \cos(\theta))}}$$

Analysis of the Set of Possible Solutions: Direction of Velocity

The case of minimum velocity at the initial point corresponds to:

$$\mathbf{d} = \mathbf{r}_2 - \mathbf{r}_1, \quad a^{\min} = \frac{r_1 + r_2 + d}{4}, \quad v_1^{\min} = \frac{2\mu}{r_1} - \frac{\mu}{a^{\min}}, \quad \operatorname{tg}(\theta_1^{\min}) = \frac{v_1^{\min} \sin(\varphi)}{2r_1(1 - \cos(\varphi))}$$

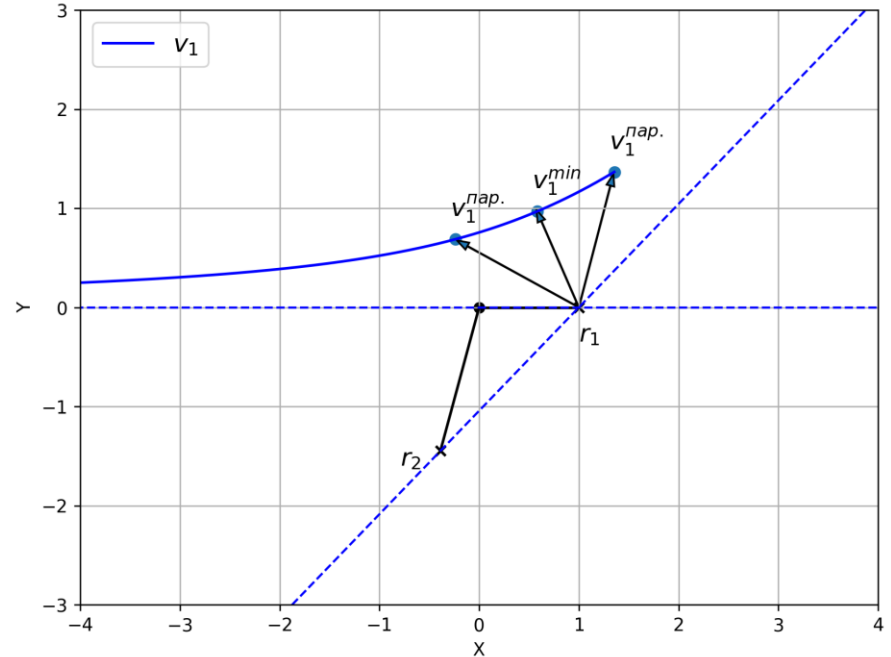
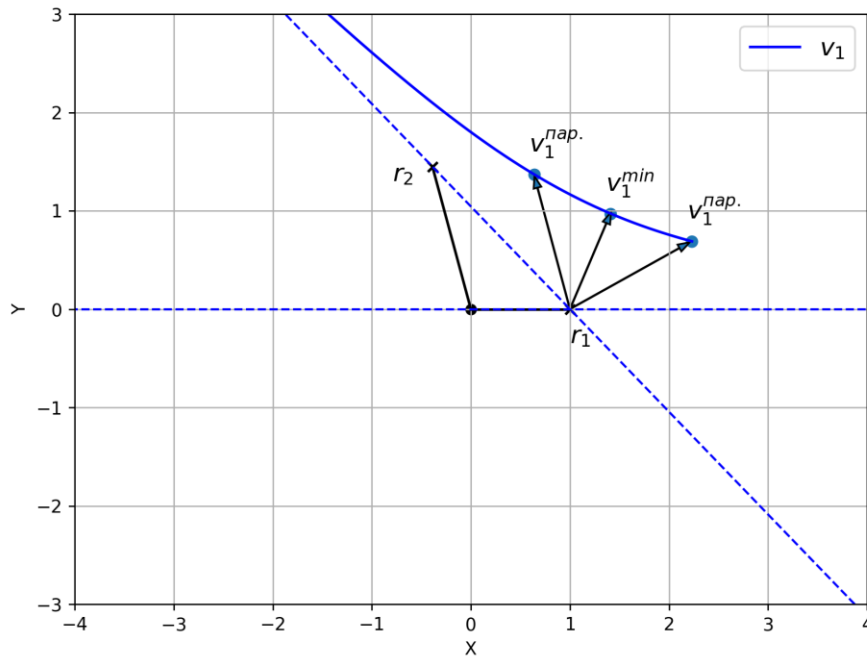
The limiting value of velocity for a hyperbolic solution corresponds to:

$$\cos(\theta_1^{\text{zun}}) = \frac{\sqrt{d^2 - (\mathbf{r}_1^T \mathbf{d})^2 / r_1^2}}{d}, \quad \text{если } \varphi < \pi \quad \text{и} \quad \theta_1^{\text{zun}} = -\frac{\pi}{2}, \quad \text{если } \varphi > \pi$$

Parabolic solutions:

$$\operatorname{tg}(\theta_1^{\text{nap}\pm}) = \frac{\sin(\varphi) \pm \sqrt{2r_1(1 - \cos(\varphi)) / r_2}}{(1 - \cos(\varphi))},$$

Analysis of the Set of Possible Solutions: Direction of Velocity



$$r_1 = 1, \quad r_2 = 1.5, \quad \varphi = \pm 105^\circ$$

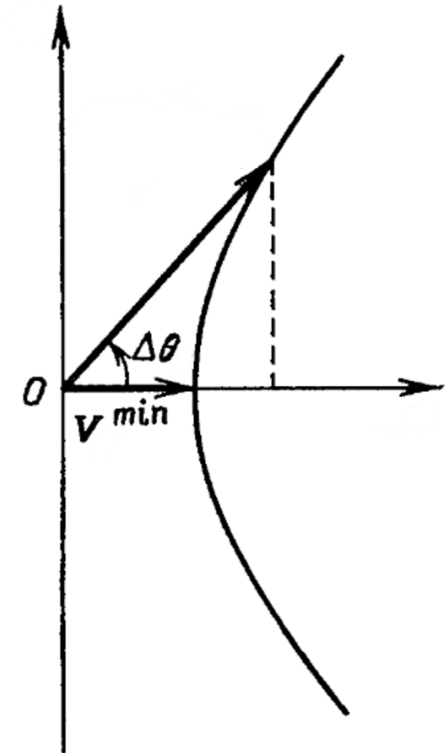
Analysis of the Set of Possible Solutions: Direction of Velocity

Solution of Clairaut's equation in Okhotsimsky's form (for the inverse problem of ballistics):

$$v_1(\theta) = \sqrt{\frac{\mu r_2 (1 - \cos(\varphi))}{r_1 (r_1 \cos^2(\theta) - r_2 \cos(\varphi + \theta) \cos(\theta))}}$$

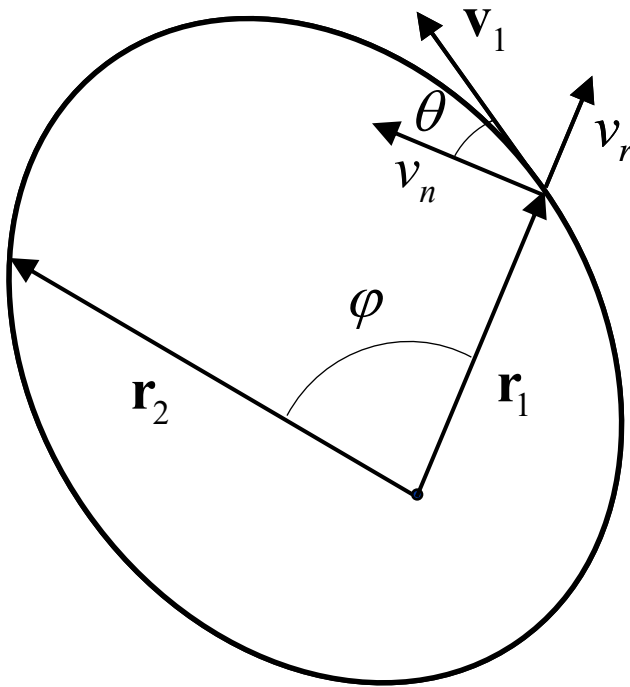
The tip of the initial-velocity vector traces a hyperbola symmetric with respect to the minimum-velocity vector; this hyperbola can be represented in the form:

$$\Delta\theta = \theta - \theta_1^{\min},$$
$$v_1(\Delta\theta) = \frac{v_1^{\min} \cos(\theta_1^{\min})}{\sqrt{\cos^2(\theta_1^{\min}) - \sin^2(\Delta\theta)}}$$



Determination of the Initial and Final Velocity Vectors

Computation of the velocity on the transfer trajectory for a given areal-velocity constant:



$$v_{r1} = v_1 \sin(\theta), \quad v_{n1} = v_1 \cos(\theta),$$

$$h = v_1^2 - \frac{2\mu}{r_1} = -\frac{\mu}{a}, \quad c = r_1 v_{n1},$$

$$f = 1 - \mu r_2 \left(1 - \cos(\varphi)\right) \frac{1}{c^2},$$

$$g = r_1 r_2 \sin(\varphi) \frac{1}{c},$$

$$\dot{g} = 1 - \mu r_1 \left(1 - \cos(\varphi)\right) \frac{1}{c^2},$$

$$\mathbf{r}_2 = f\mathbf{r}_1 + g\mathbf{v}_1, \quad \mathbf{v}_2 = \dot{f}\mathbf{r}_1 + \dot{g}\mathbf{v}_1,$$

$$\mathbf{v}_1 = \frac{\mathbf{r}_2 - f\mathbf{r}_1}{g}, \quad \mathbf{v}_2 = \frac{\dot{g}\mathbf{r}_2 - \mathbf{r}_1}{g}.$$

Determination of the Transfer Time

To compute the transfer duration "without" Kepler's equation, using an explicit dependence on the initial velocity v_1 and the flight-path angle θ :

$$\Delta t = \frac{r_1}{v_1 \cos \gamma} \left[\frac{\tan \gamma (1 - \cos \varphi) + (1 - \lambda) \sin \varphi}{(2 - \lambda) r_1 / r_2} + \frac{2 \cos \gamma}{\lambda (2 / \lambda - 1)^{3/2}} \operatorname{arctg} \left(\frac{\sqrt{2 / \lambda - 1}}{\cos \gamma \operatorname{ctg}(\varphi / 2) - \sin \gamma} \right) \right].$$

where:

$$\lambda = \frac{r_1 v_1^2}{\mu}, \quad \gamma = \pi - \theta.$$

Determination of the Transfer Time

Computation of the transfer duration using Kepler's equation in Lagrange's form:

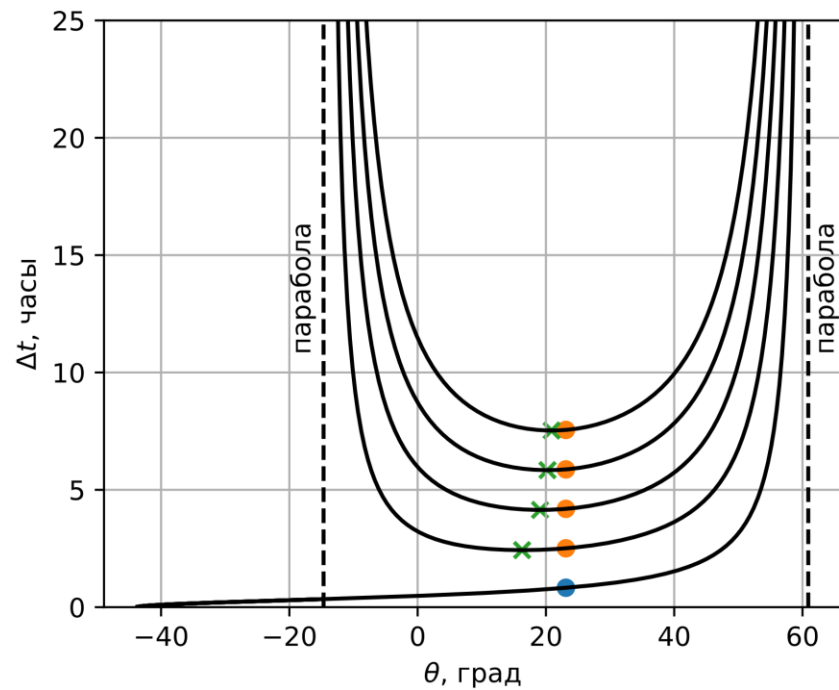
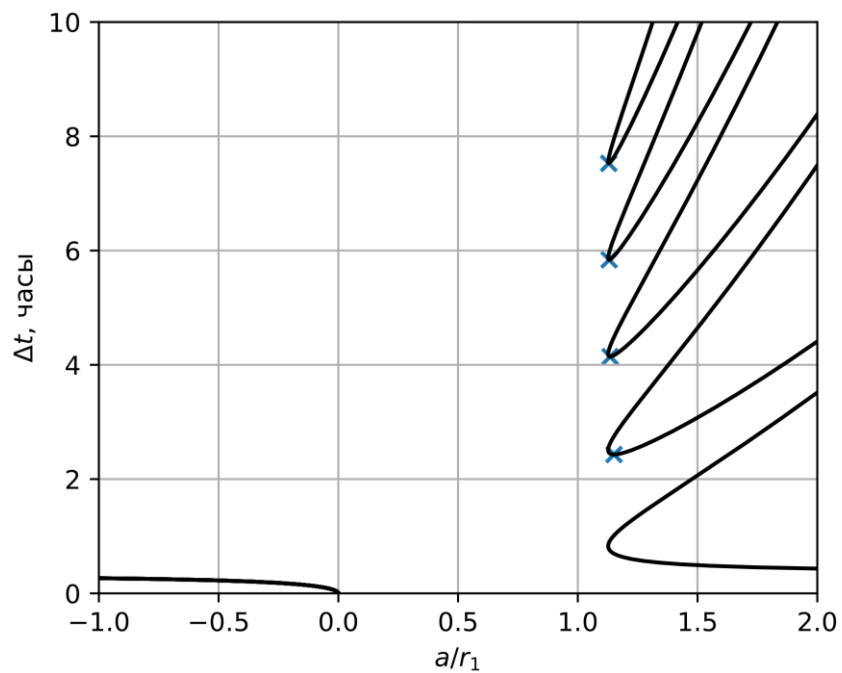
$$\mathbf{d} = \mathbf{r}_2 - \mathbf{r}_1, \quad \mathbf{u}_1 = \frac{\mathbf{r}_1}{r_1}, \quad \mathbf{u}_2 = \frac{\mathbf{r}_2}{r_2}, \quad \mathbf{u}_d = \frac{\mathbf{d}}{d},$$

$$\begin{aligned} \mathbf{v}_1 &= (B + A)\mathbf{u}_d + (B - A)\mathbf{u}_1, \\ \mathbf{v}_2 &= (B + A)\mathbf{u}_d - (B - A)\mathbf{u}_2, \end{aligned} \Rightarrow A = \frac{v_d - v_r}{2}, \quad B = \frac{v_d + v_r}{2},$$

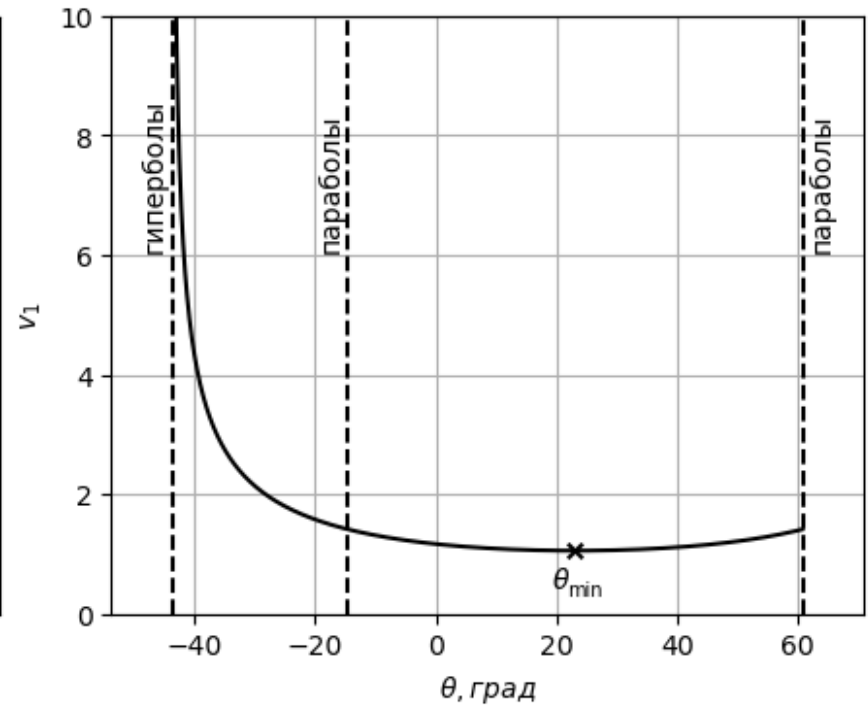
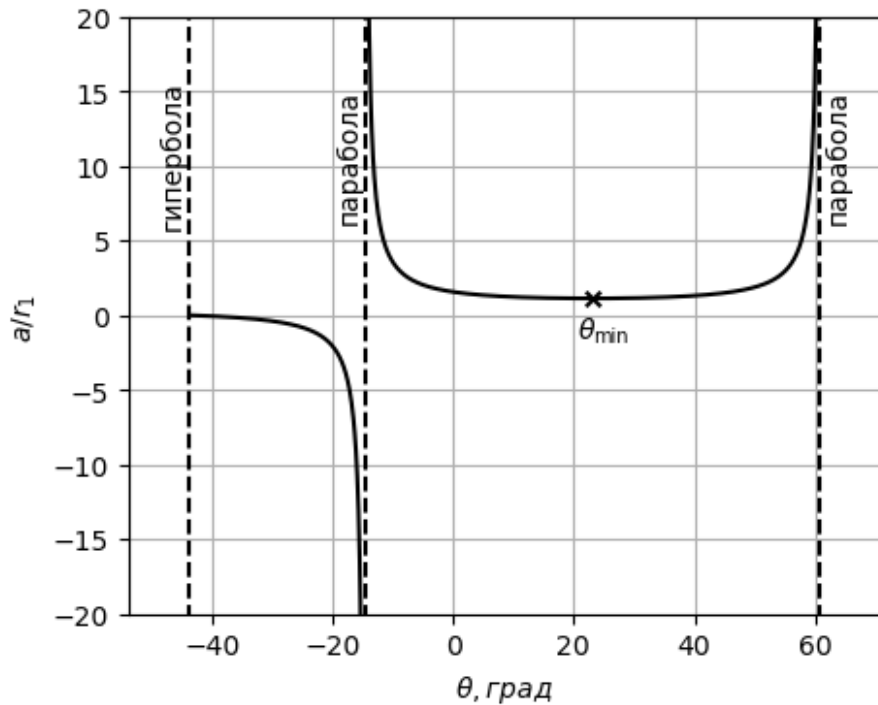
$$\sin\left(\frac{\alpha}{2}\right) = \sqrt{\frac{s}{2a}}, \quad \cos\left(\frac{\alpha}{2}\right) = A\sqrt{\frac{2s}{\mu}}, \quad \sin\left(\frac{\beta}{2}\right) = \sqrt{\frac{s-d}{2a}}, \quad \cos\left(\frac{\beta}{2}\right) = B\sqrt{\frac{2(s-d)}{\mu}}$$

$$\Delta t = \sqrt{\frac{a^3}{\mu}} (\alpha - \sin(\alpha) - \beta + \sin(\beta) + 2\pi N)$$

Analysis of the Set of Possible Solutions



Analysis of the Set of Possible Solutions



(independent of the number of revolutions)

Estimate of the Convergence Rate

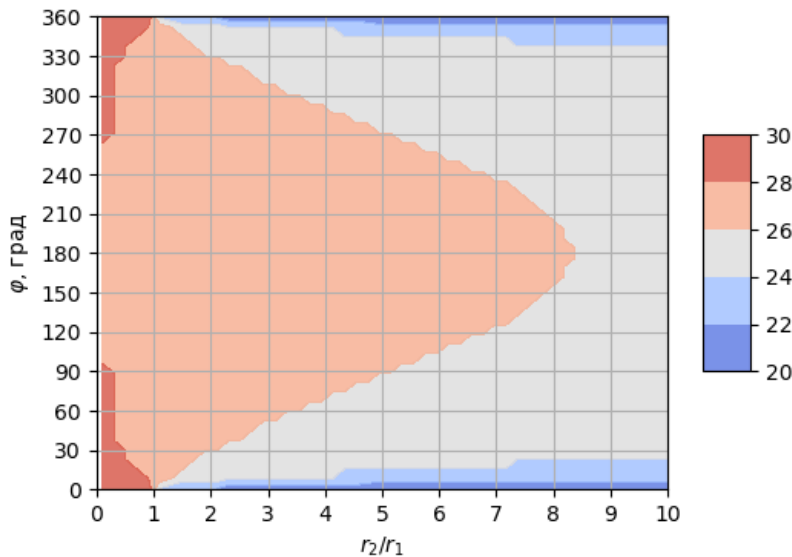
Let us estimate the maximum number of iterations required when using the **bisection method**:

- the initial bracketing interval for the root, for elliptic and hyperbolic solutions

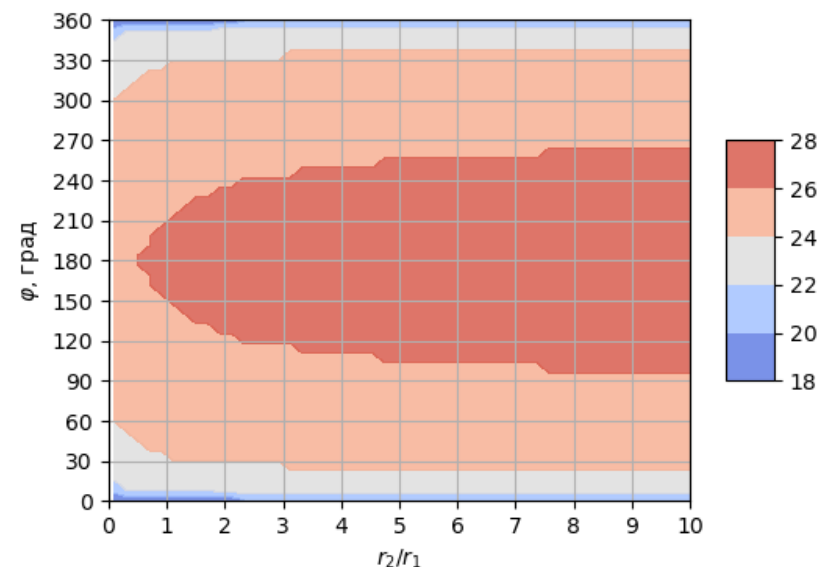
$$\theta_1 \in [\theta_1^{zun}, \theta_1^{nap-}], \quad \theta_1 \in [\theta_1^{nap-}, \theta_1^{nap+}]$$

- an estimate of the number of iterations as a function of the length of the initial bracketing interval and the required accuracy

$$n = \log(|\theta_1^2 - \theta_1^1|/\varepsilon) / \log(2)$$



Elliptic solutions



Hyperbolic solutions

Solution Algorithm (Halley's Method)

To solve the resulting nonlinear equation, we apply

$$\Delta t(\theta, N) = \Delta t^*,$$

Halley's method, whose iteration takes the following form

$$\theta^{i+1} = \theta^i - \frac{2\Delta t(\theta^i)\Delta t'(\theta^i)}{2\Delta t'^2(\theta^i) - \Delta t(\theta^i)\Delta t''(\theta^i)},$$

where the following expressions can be used to compute the first and second derivatives

$$\Delta t'(\theta^i) = \frac{d\Delta t}{da} \frac{da}{dv_1} \frac{dv_1}{d\theta}, \quad \Delta t''(\theta^i) = \frac{\Delta t'(\theta^i) - \Delta t'(\theta^{i-1})}{\theta^i - \theta^{i-1}}$$

$$\frac{d\Delta t_{\text{ell}}}{da} = \frac{3}{2} \sqrt{\frac{a}{\mu}} \left(2\pi N + (\alpha - \sin(\alpha)) - (\beta - \sin(\beta)) - \frac{4\sin^3(\alpha/2)}{3\cos(\alpha/2)} + \frac{4\sin^3(\beta/2)}{3\cos(\beta/2)} \right),$$

$$\frac{da}{dv_1} = \frac{2\mu v_1}{(-2\mu/r_1 + v_1^2)^2}, \quad \frac{dv_1}{d\theta} = \frac{v_1 \sin(\Delta\theta) \cos(\Delta\theta)}{\cos^2(\theta_{\min}) - \sin^2(\Delta\theta)}.$$

Solution Algorithm (Halley's Method)

Based on the analysis of the solution set, an effective rule for generating initial guesses can be constructed, based on approximating the dependence of the transfer time on characteristic reference points.

Initial guess for **Halley's method**:

- for **hyperbolic** solutions (linear approximation)

$$\theta_{\text{hyp}}^0 = \theta_{\text{lim}} + \frac{(\theta_{\text{par-}} - \theta_{\text{hyp}})}{\Delta t_{\text{par}}} \Delta t;$$

- for **elliptic** solutions, when the transfer arc spans less than one revolution $N = 0$ (hyperbolic-type approximation)

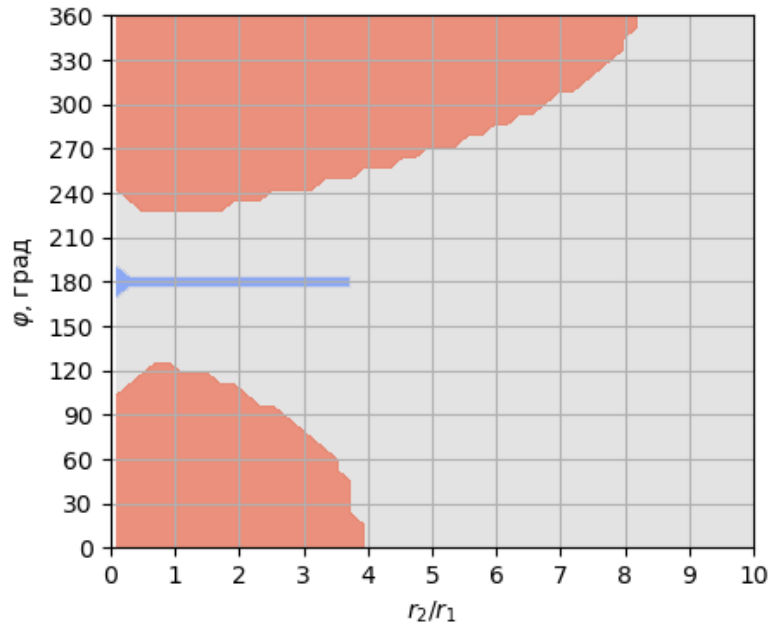
$$\theta_{\text{ell}}^0 = \theta_{\text{par+}} - \frac{k}{\Delta t - b}, \quad k = \frac{\Delta t_{\text{min } a} - \Delta t_{\text{par}}}{1/(\theta_{\text{par-}} - \theta_{\text{par+}}) - 1/(\theta_{\text{min } a} - \theta_{\text{par+}})}, \quad b = \Delta t_{\text{par}} + \frac{k}{\theta_{\text{par-}} - \theta_{\text{par+}}};$$

- for multi-revolution **elliptic** solutions $N > 0$ (parabolic-type approximation)

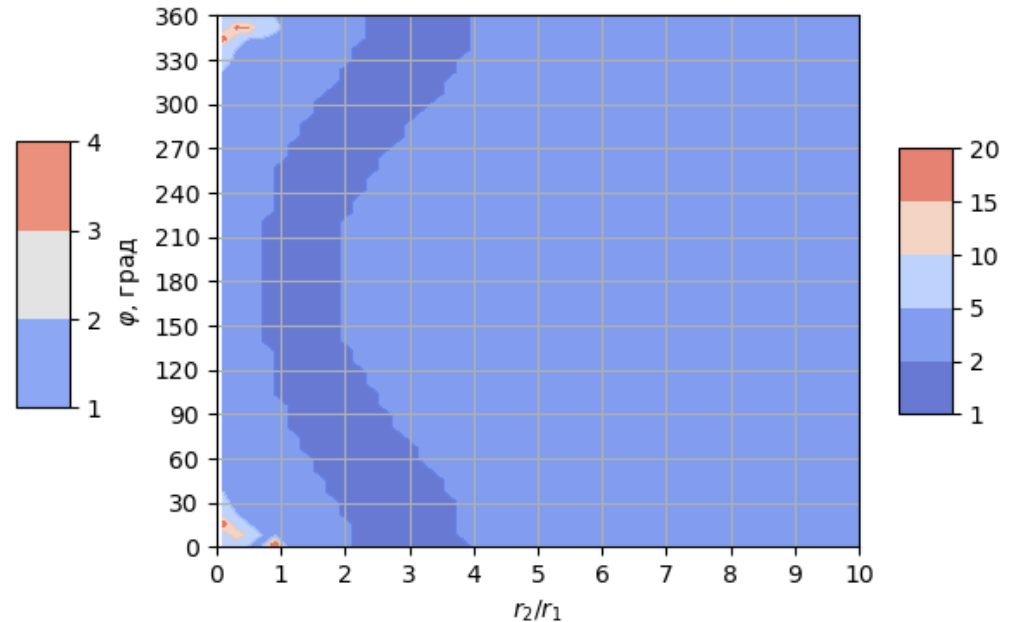
$$\theta_{\text{ell}N-}^0 = -\sqrt{\frac{\Delta t}{10\Delta t_{\text{min } N}}} - 0.1(\theta_{\text{min}} - \theta_{\text{par-}}), \quad \theta_{\text{ell}N+}^0 = \sqrt{\frac{\Delta t}{10\Delta t_{\text{min } N}}} - 0.1(\theta_{\text{par+}} - \theta_{\text{min}}).$$

Estimate of the Convergence Rate

Let us estimate the maximum number of iterations required when using **Halley's method**.



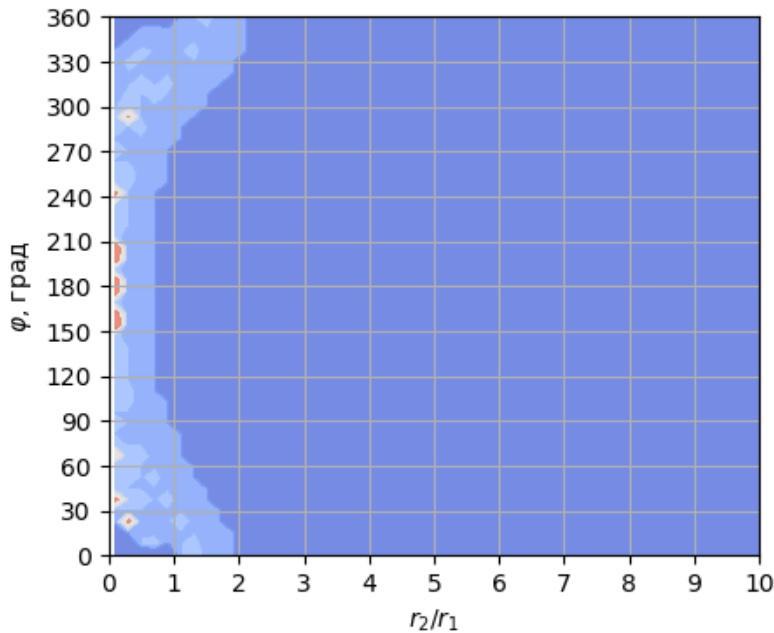
Elliptic solutions ($N = 0$)



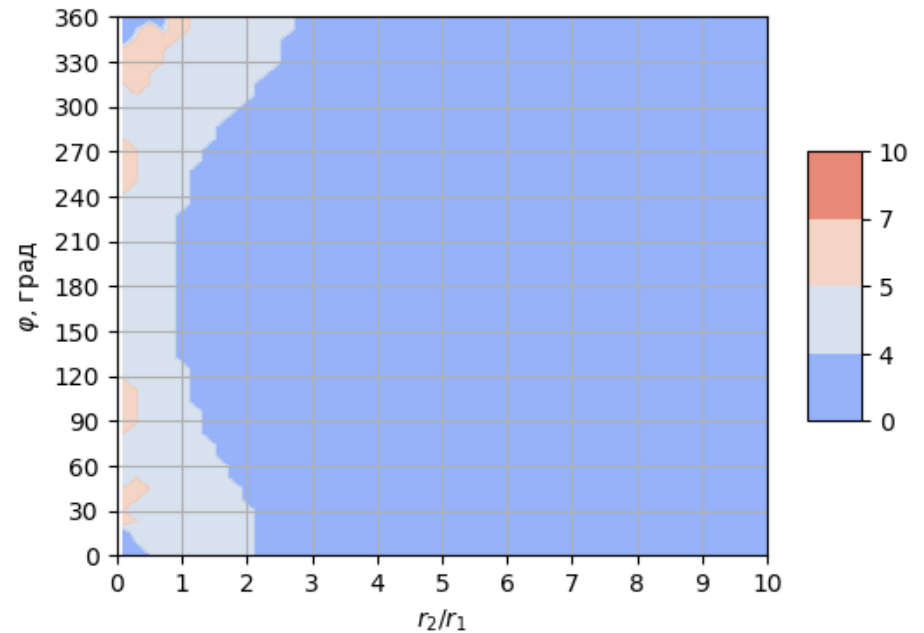
Hyperbolic solutions

Estimate of the Convergence Rate

Let us estimate the maximum number of iterations required when using **Halley's method**.



Elliptic solutions
($N = 1$, left)



Elliptic solutions
($N = 1$, right)

Conclusion

We have considered a solution based on **the ballistic approach of Okhotsimsky** and the solution of Clairaut's equation for the Kepler problem. A **qualitative analysis** of the solutions has been carried out. **According to the estimated convergence rate for the solution of the nonlinear equation, the typical number of iterations is on the order of 5.**

Advantages of the presented method:

- ❖ overall **transparency** of the algorithm;
- ❖ explicit dependence of the solution on the **initial velocity vector** (which allows it to be efficiently incorporated into trajectory-optimization procedures).

Prospects for further development:

- development of **trajectory-optimization methods** based on the solutions considered;
- accounting for **perturbations** possessing invariants analogous to the energy constant (for example, the J2 zonal harmonic, etc.).

Ivashkin V.V. // **Doklady RAN. Physics. Technical Sciences. 2024. Vol. 514. P. 58–62.**

Ivanyukhin A.V., Ivashkin V.V. // **Ivanyukhin, Ivashkin // Solar System Research, 2024. Vol. 58. No. 6. P. 771–782.**

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Thank you for your attention

